Exploring the Student’s Conception of Mathematical Truth in Mathematical Reasoning

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Introduction

Since the recent inclusion of mathematical reasoning as an aspect of student’s mathematical competence, the development of students’ mathematical reasoning has become one of the main issues in mathematics education (NCTM, 1989, 1991, 2000; Stiff & Curcio, 1999). From a cognitive standpoint, students’ processes of reasoning can be influenced by their embodied experiences of mathematical objects and representations (Thompson, 1996); their problem solving expertise (Polya, 1954); the levels of sophistication in their proof schemes (Balacheff, 1988; Sowder & Harel, 1998); and their interpretations of logical implications (Durand-Guerrier, 2003; Hoyles & Küchemann, 2002).

Research studies on students’ proof schemes and conception of logical implications have revealed that individual’s cognitive processes of applying mathematical logic to mathematical situations are subjected to several considerations (Balacheff, 1988; Duval, 1991; Healy & Hoyles, 2000; Hoyles & Küchemann, 2002; Segal, 1999). Across these lines of inquiry, the students’ conceptions of mathematical proofs and logical implications are identified as two cognitive factors in their transition from an informal type of reasoning towards a more formal mathematical. Moreover, the students’ views and understanding of proofs and reasoning varied and do not conform to a logico-mathematical norm at large (Durand-Guerrier, 2003; Healy & Hoyles, 2000; Hoyles & Küchemann, 2002).

What has escaped attention in these inquiries is students’ conceptions of mathematical truth. Some research reports have suggested that the truth value of a mathematical statement for students may be dependent upon the existence of a mathematical proof or the semantics of logic. For example, pre-service teachers (college students in education classes) reversed their conclusions about the mathematical truth of the statement when they agree that an intuitively irrelevant case could satisfy a conditional "if-then" implication logically (Durand-Guerrier, 2003). About half of the students in an assessment study viewed proof as a means to establish mathematically true statements (Healy & Hoyles, 2000). These reports of students' mathematical reasoning suggest that students' conception of truth may be another possible cognitive factor in students' mathematical reasoning processes. Given the findings that students' interpretations of logical implications and conceptions of mathematical proofs are either inconsistently applied or deviant from the norms of mathematical communities at large, isn't it possible that students' conceptions of "true" might deviate widely too? To extend the same query to the students' conception of mathematical truth seems like a plausible research stance.

Our study thus concerns with this additional complexity behind student's development of mathematical reasoning and aims to understand the students' conception of mathematical truth as our objects of investigation. One way of beginning this line of inquiry is by taking a step back and using case studies of students' mathematical reasoning to inquire whether students' conceptions of mathematical truth concur with the logico-mathematical standard. In addition, the influences of students' mathematical truth conceptions over their validations of mathematical statements are explored.

Theoretical framework

Our theoretical framework stems from three theoretical considerations: the normative conception of mathematical truth, the views of (mis)conceptions, and parameters of human reasoning processes. In this section we will lay out these considerations that afford our theorization of mathematical reasoning and our interpretations of the results of this study.

Views of Mathematical Reasoning

The view of mathematical reasoning in this study leans towards the cognitive perspective and is thus defined as “a directed, purposeful form of thinking in which additional mathematical information is derived from given information in order to draw a valid mathematical conclusion”. This definition can be taken as an information-processing view of human reasoning processes in the context of establishing mathematical knowledge (Evans, 1993).

The mathematical context of this definition resembles prior studies of students’ proof schemes (Balacheff, 1988; Harel & Sowder, 1998) and interpretation of logical implications (Durand-Guerrier, 2003; Hoyles & Küchemann, 2002). In these studies, the context calls for students to examine and establish the assertions of mathematical statements within given constraints in an individual setting. Their conceptions of mathematical truth are employed while they accomplish these authentic tasks.

Normative conception of mathematical truth

The notion of the normative conception of mathematical truth adopted in this study is similar to the logicist frame of mathematical truth. By this, we meant the normative conception to be adhering to the truth definitions imposed by classical propositional and first-order logic, whereby the law of the excluded middle states that the negation of “true” is “false”, and vice versa. Such a notion is still widely employed in mathematics text and practices as seen in the proof of the irrationality of $\sqrt{2}$ or the infinite numbers of primes. Our adoption of this logicist frame serves more of an analytic and practical purpose in our study than an epistemic endorsement. The latter belongs to the intellectual work of the philosophers of mathematics and logic.

Under the classical propositional and first-order logic, each mathematical statement is assigned a binary truth value of “true” or “false”. The negation of “true”, or “Not true”, is equivalent to “False” and vice versa. The truth value of each mathematical statement is assigned according to Tarski's principle of compositionality: the logical value of a mathematical statement is a recursively defined by the logical composition of its components (Hintikka, 1996). The assignments of truth values are consistently applied to mathematical statements of the same logical structure regardless of their mathematical purposes and contexts. Thus, the truth values we assigned to logical conjunction (reasoning about the "AND" operator in a mathematical statement) of different mathematical statements are consistent with the truth table of a logical conjunction (Benacerraf & Putnam, 1964; Hintikka, 1996).

Psychological views of "conceptions/misconceptions"

It is a common phenomenon in the process of learning that students construct their own conceptions about objects and their properties (Smith, diSessa, & Roschelle, 1993). In math and science learning, such conceptions are prevalent among students (Reiner, Slotta, Chi, & Resnick, 2000). They are sometimes referred to as prior conceptions, alternative conceptions, naïve conceptions, and have been recently referred to as misconceptions (Chi, 2005; Hardy, Jonen, Moller, & Stern, 2006; Smith et al., 1993).

Current research studies of students' misconceptions generally attributed the underlying causes to students' cognitive characteristics and resources based on different and competing theoretical frames (Chi, 2005; Smith et al., 1993; Vosniadou & Brewer, 1992). Despite their differences, a common theme that runs across these theoretical frames is that students can apply different and often contradictory conceptions to situations that are equivalent discipline wise. This theoretical consideration hence implies that students may apply the normative conception as well as their own conceptions of mathematical truth to different types and contexts of mathematical reasoning tasks. Such an implication provides an avenue of incorporating and interpreting students' thoughts and conceptions which may seem illogical and un-mathematical initially.

Conceptions of Mathematical Truths as underlying notions of reasoning tasks

In any reasoning task, one has to interpret the given conditions through some representational systems and the notion of a valid conclusion as presented by the task. Thus, his/her reasoning process can be further subdivided into two distinct sub-processes: reasoning for an interpretation of the task and reasoning for a conclusion based on these interpretations (Stenning & Lambalgen, 2004; Stenning & Monaghan, 2004). The former process sets the notion of valid conclusion towards which one’s reasoning processes are driven while the latter process generates inferences and conclusions strategically according to his/her interpretations and notions. This distinction helps to explain the student’s mathematical reasoning process in the light of his/her own interpretation of the task. Students’ notion of “true”, “false” and validity are thus considered as their interpretations of the reasoning task which set the parameters of their reasoning and guides their subsequent reasoning process.

Stenning & Lambalgen (2004) have identified subjects’ conception of truth and falsity in deductive reasoning as influential factors of their reasoning processes and conclusions made in an abstract context, e.g., some subjects thought that a logical implication absent of counterexamples did not compel them to conclude it to be “true”. We extended their theoretical frame to the mathematical context of this study. That is, students’ conceptions of mathematical truths were considered as the students’ “reasoning for interpretation” processes from which they subsequently reasoned to decide the validity of mathematical statements.

In order to elicit such mathematical reasoning processes from students, we chose reasoning tasks that required students to validate a mathematical assertion posed in each task. In each task, a mathematical situation was set up with a fictional character posing a mathematical assertion. The entire collection of tasks contained a mix of various mathematical context such as numerical, algebraic, function and graphical.

Having these tasks set up as outlined above, our research questions are thus as follows: 1) What are students' conceptions of mathematical truth? 2) What cognitive roles do their conceptions of mathematical truth play in their mathematical reasoning?

Method

The data used in this study came from an exploratory study of college students' reasoning processes. Two mathematics major and four non-mathematics major college students from a large Midwestern university have volunteered to participate in the study. Of the 2 mathematics majors, one was a freshman and the other a senior. Both had taken AP calculus during their high-school years. At the time of interview, the non-mathematics majors were taking or had recently completed a college algebra class while the mathematics majors had completed a calculus class. Almost all of them had taken 4 years of typical high school mathematics classes such as Algebra I and II, Geometry, Trigonometry and Statistics or Calculus. The diversity in this sample shown in the table below exhibited variability required for the exploratory purpose of the study here.

Table 1*: The mathematical background of the participants*

Name	College Status	Major	College Math Experience	High School Math Experience	Current Math Class	Time of Interview
Lisa	Freshman	Political Science or Business	College Algebra	4 years with Trigonometry and Statistics	College Algebra	Jun 06
Polly	Freshman	Medical Sciences	College Algebra	4 years with Trigonometry and Statistics	College Algebra	Jun 06
Sharon	Freshman	Non-Math; Undecided	College Algebra	3 years, no Pre-Calculus	None	Jun 06
Darren	Sophomore	Supply Chain/ Finance	Statistics II; College Algebra	4 years with AP Calculus	College Algebra	Jun 06
Pam	Freshman	Math	Applied Calculus; Calculus I	4 years with AP Calculus	Calculus I	Nov 06
Winnie	5th year Senior	Math	Typical Math major courses	3 ½ years with AP Calculus	Complex Analysis	Feb 07

Brief profile of each student

Lisa. At the time of interview, Lisa had just completed her freshman year and was considering political science or business management as her major. To meet the university's mathematics requirement, she took a college algebra course. She chose to do it during the summer because the assessments were given in the short answer questions format instead. During high school, she had four years of a typical combination of mathematics classes (Algebra I, Geometry, Algebra II, Trigonometry and Statistics) and gained an average grade of B or B minus.

Polly. At the time of interview, Polly had just completed her freshman year and was considering medical sciences as her major. In the summer, she took her college algebra course to fulfill the university's requirement. During high school, she had four years of a typical combination of mathematics classes (Algebra 1 and 2, Geometry and Statistics) and gained grades between A's and B's. She felt that mathematics was easy but became more difficult during high school years.

Sharon. Sharon is a non-math major freshman whose major was not made known to the author then. At the time of interview, Sharon had already completed her college algebra course with a 3.5 grade a semester ago. She was placed into the intermediate algebra course by the college's math placement test but she did not want to spend an extra semester in taking a course that would not contribute to her college credits nor fulfill her math requirement for all non-math majors. Instead, she managed to enroll into the college algebra. During her high school years, she took a math class for each year (Algebra I, Geometry, and Algebra II) but skipped math in her senior year because she did not want to take Pre-calculus. She found Geometry class difficult with so many theorems to remember and prove and got a B grade. In comparison, the Algebra classes were much easier and she got A grades for both.

Darren. Darren was a sophomore majoring in supply chain management and finance. At the time of interview, he had almost completed his college algebra course. He took this class to obtain some effortless credits to fulfill his elective requirements so he can concentrate more of his resources towards his majors. In fact, he was over-qualified to take this college algebra course since he had already completed an AP calculus with a score of 4 points during high-school. Besides AP calculus, David had taken Analysis, Geometry and Algebra, one for each year of high-school. He got grades A’s and B’s for these math classes. Up to the time of interview, he had already taken two college math classes, Statistics and College Algebra, of which he obtained a 3.5 grade for the former and was confident of a 4.0 grade for the latter.

Pam. At the time of interview, Pam was in her second semester of her freshman year in math major. She was taking her second math class, Calculus I, after she had completed an Applied Calculus course in the previous semester. Before she came to college, she had already taken 6 math classes in her high school and a local community college. These classes included AP Calculus in her high school and Trigonometry and Statistics in a local community college. She had attained an average of A’s and low A’s for all these classes. She was slightly dyslexic but this did not affect her reasoning of the task posed much since she was able to correct herself and understand the actual task demands when she re-read the question later.

Winnie. Winnie was a senior math major taking some teacher education courses in order to become a qualified high school math teacher. At the time of interview, Winnie was taking complex analysis and an education class that required her to observe teaching in a local high school algebra classroom twice per week. Besides observing, she would also help the students with their group work. She had also some tutoring experiences during her college years. Winnie completed her Algebra I course during her middle school years. During her high school years, she spent three and a half years taking all math programs available to her then, which includes Algebra II, Pre Calculus, Trigonometry and AP Calculus. Her grades were mostly A and A minus, except for her AP calculus, which she scored a 3 point. Due to her teacher training and math training, she had a lot of experiences in dealing with Algebra two contents, functions and graph, and could remember most of the contents when she was attempting the reasoning tasks.

Data collection process

Students met individually with the researcher for approximately 90 minutes and were given a pre-task survey and a think-aloud task in which the participant worked alone, followed by a semi-structured interview with the researcher. During the think-aloud and interview session, the students' verbal utterances and reasoning activities were video-recorded for data analysis purposes.

The tasks

The pre-task survey had 18 items which collected each student's level of conviction about each assertion by placing a mark on a linear continuous scale ranging from "true" at the left end, through "I don't know" and then to "False" at the right end. This information was then used for identifying 4 tasks of mixed levels of conviction to be used in the think-aloud task session. The student was then given these four tasks and was left alone to think-aloud and write down his/her response as he/she established the validity of the assertion in each task phrased neutrally in the voice of a character. Each student was then interviewed for elaborations about his/her work and his/her interpretation of the task, including his/her view of true/false mathematical statements.

Based on the video, verbal and written data, a case analysis was completed for each student. First the students' interview data were examined to identify episodes where they spoke about the truth value of mathematical assertions and how they established truth/falsity. From these data, specific attention is given to the analysis of the range of "truth values" (e.g., "true", "false", "mostly true", not "true" will be "false") in students' conception and the rationale underlying their validation. Findings were then generated from the variations in these attributes and comparisons made to the normative conception of truth and the processes of classical mathematical logic. These comparisons revealed to us the characteristics of these variations from the normative conception.

Results

In this section we present a 'distance' model of the students' conception of mathematical truth which best describes the variability seen in our data. The details of the variations will be laid out in clearer terms as we refer to the interview data later. In addition to that, we will present data that show the influence of students' conceptions on their mathematical reasoning processes seen at the level of students' assignment of mathematical truth values to a mathematical assertion.

Distance model of students' conceptions of mathematical truth

The students' conceptions of mathematical truth deviated at different degrees from the normative conception of mathematical truth. Of the three students with AP Calculus background, their conceptions resembled closely to the normative conception. Of the other three non-AP students, Sharon's conception was the closer, followed by Polly's that had an sense of uncertainty, and Lisa's which deviated radically and would be deemed as "un-mathematical" or "illogical". Hence each student's conception can be positioned on a 'distance' model as follows:

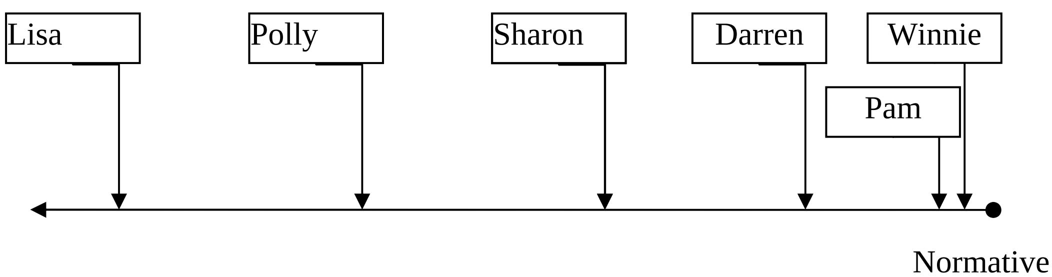

Figure 1: Distance model of the students' conception of mathematical truth

The intervals in this model between each position depict the qualitative differences as compared to the attributes of the normative conception. A student's conception that exhibits more qualitative variations along several attributes is placed "further" away from the normative end. One quick observation contrast can be made is the divergence of the conceptions (from the normative) among the students with a "standard" high school program but without Calculus as compared to the closer alignment of conceptions among the students who took Calculus in high school. This association sheds no direct light on whether it is the course *per se* (AP Calculus) or the students who enroll in the course that most affected the observed conception of truth.

Students' Conceptions that resembled the Normative Conception

The three students with AP Calculus background whose conceptions were very much aligned with the normative conception. Winnie, Pam and Darren all asserted that a mathematical statement is either "true" or "false". They also conceived a binary relation between the truth values "True" and "False" which resembled the law of the middle exclusion in classical logic: a statement that is not "true" is "false", and vice versa. To them, "True" is assigned to a mathematical statement only when every single case of the statement must fit what the statement had asserted or the statement would be "False". Their conceptions were consistently applied across different mathematical tasks and contexts. In addition, Winnie and Pam referred to a standard, human-error-free practice of demonstrating the statement's truth value in which themselves can get involved. This was understandable given that these math practices are common in their advanced college math classes.

Students' Conceptions that resembled the Normative Conception

Of the three students who did not take AP Calculus, Polly and Lisa exhibited conceptions of mathematical truth that were most different from the normative conception. Lisa's conception deviated to the extent of being illogical or "un-mathematical" in the eyes of the mathematical discipline.

Polly - There is always an exception. Polly held a conception of mathematical truth that bore some resemblance to the normative conception. Like the other four who held normative conception, she asserted that a statement is either "true" of "false" and that a case that did not work for a general statement would render it as "false".

Excerpt 1: Polly – it’s either right or wrong

Polly: Oh yeah, it's either like you have *right* or *wrong*, and then it's like, oh here, there is an *exception*. What do you mean there's an *exception*? That's *not supposed* to happen!

I: Exception? So, uh, can you give me an example?

Polly: Well, when it's like *two plus two is four unless there is an imaginary number*, I thought I’m like, Oh, So it's not like four anymore, like [throwing her both hands slightly up into the air] then what? Two plus two is not four? And then I get really confused.

I: Unless there is an imaginary number?

Polly: Yeah, there was like an imaginary, when there is the imaginary number you can throw it and then change the answer or

I: yeah

Polly: or there's an, I don't know

To Polly, every mathematical statement can only be either “True” or “False” which was evidenced by its outcome of “Right” or “Wrong”. However, she did not believe that the assignment of a truth value to the statement was permanent because there were always some inexplicable constraints that would render the statement “wrong” beyond her understanding. Such a sense of uncertainty was evident in her discussion of a fundamental addition fact 2 + 2 = 4. Her assignment of “true” values was psychologically unstable. This was evident in the way she considered the assignment of “true” to a mathematical statement.

Excerpt 2: Polly – not 100% true but not in the middle either

P: But, I don't know. I think the true and false is good but I didn't know where to put it [points to the linear continuous scale of "true" and "false"] on the line, like, cuz it is not 100% true, but I don't know if I want to put it in the middle either.

I: So when you say, when you put something, well like, somewhere close to a hundred but not exactly at 100%, what are you trying to say?

P: Well, I was like I was pretty sure, like it sounded true, but then again I am like, maybe I don't have any idea what I am talking about so maybe I should like move it over here [point toward the middle of the linear scale] a little bit.

I: ok, and, so when people say that in mathematics, this is true, um, what do you make of that sentence?

P: Or, but I feel like this problem is true but if you do it again with a different number, then there is an exception to the rule that may be false.

I: Ok, and so when people say something like, in mathematics this is true, um, you would think that there still might be some exception?

P: right.

Polly consciously refrained from stating "true" in absolute degree. One could then conclude that the epistemic values she held for most mathematical statements was not synchronized with the logical values, as a result of the sense of uncertainty in her conception (Duval, 1991). It was also clear from her descriptions that changing the numerical value of the mathematical statements can affect its truth value. This was a deviation from the normative conception. A mathematical statement may change its truth value due to an extension of its conditions or the definitions of its terms, as seen in the progress of mathematical knowledge. The original mathematical statement would still hold true regardless of the particular numerical values it takes.

One of her tasks was to determine if the following assertion was true:

> "The graphs of a quadratic function $f(x)$ and a linear function $g(x)$ passes through the points (3, 2) and (7, 5). Alan says that the graph of f is above the graph of g because the power of x in function $f(x)$ is higher. Do you think what Alan says is true? Please state why."

Polly's written response was, "Yes, Alan appears to be right, but not 100% positive because I don't remember how to graph the functions." Her subsequent explanations revealed the sense of uncertainty in her conceptions of "true" mathematical statements.

Excerpt 3: Polly – I didn't know what the exception was but it's there

Polly: yeah, well when I had read it, I kind of felt like the quadratic function is the better that I knew, like I was more comfortable with the quadratic function. So it's that I guess that could be right. Like I am still felt like I *wasn't completely sure*, like it wasn't. Well, there could be an *exception to the rule*.
I: So why weren't you completely sure? Is it because you are worrying that there is an exception to the rule?
Polly: yeah
I: ok, and
Polly: or like I would think that like I didn't remember everything that have to do with both functions, then it's like, oh maybe, is it *there's some way* that I just, don't remember it, or *something*.
I: mm hmm, and do you have a way to find out what are the possible exceptions?
Polly: nope.
I: So you just feel that there might be some exceptions
Polly: yeah. Well like, it was feel like I *don't remember everything* there is to remember about, whatever.
I: and were you able to, on the spot, come up with something, meddle with some things like graphs or whatever

Polly: yeah
I: or figures to come up with exception or. Were you trying to *look for exceptions*?
Polly: *No.*
I: No.
Polly: *No, I wasn't*
I: So you feel that there might be exceptions but you weren't
Polly: I wasn't sure and I didn't know like what it was.
I: um, you didn't know what it was
Polly: like I *didn't know what the exception was*, and that was why.

Polly’s sense of uncertainty about the “true” statements was strongly held and inherently widespread. Her references to unknown exceptions of “true” mathematical statements in a variety of examples surfaced repeatedly from plain numerical facts and algebra problems to graphs of functions as seen in her interview and written work.

Lisa - “True” means “mostly true”. Lisa’s conception could be described as the most baffling to any mathematicians or mathematics educators. Lisa held two conflicting conceptions in mind: one that resembled the normative version and one that was coined as “true” means “most parts of the statement are true”.

Excerpt 4: Lisa – pretty standard conception but a little “false” should still be “true”

I: Ok, Now, when you look at questions like this, um, when you see the word true, or false, in mathematics what does that mean to you?
L: It's gonna be like, the whole thing has to be right, and that one thing is wrong, then it's got to be false, which kind of throws me off, because I'm like, wow, this part if it is true, then, like for multiple choice and stuff, I never know that if a part of it is true and then I'm like, well, I’m like, yeah, I don't, I don’t like true and false[laughs].
I: So, so you were saying that, um, the whole thing, meaning the whole statement?
L: right,
I: and it has to be true for every part?
L: right, or it's confusing because like if, every part of it is true except for one little word, and then it's like, do I say false or probably this other stuff is true. so, yeah.
I: Ok, so, what, what would be your, um, reaction to that? would you want to say true or would you say false?
L: I’d probably say true
I: I see.
L: You know, because I, I really feel like I should say true because well, such much already of it is true, and then just because like one little things is off, then [shrugs her shoulders].

She recognized that the normative conception would require her to assign "false" to a statement if one case of the statement contradicts the general statement. However, she asserted that the statement should be true if most parts of the statement were "true". Though she was aware of her noncompliance of the mathematical standard and her alternative conception might suffer some disapproval, she was confused by the rationale behind the standard conception for this type of reasoning.

Her noncompliance to the normative conception was evident when the statement required her to reason conjunctively (as in an "AND" statement). At one time during the interview, such an alternative conception reappeared when she was explaining her written work in a context of reasoning with graphs.

Lisa was looking at an instance of a newly defined class of quadratic curves:

> "A 'Tullio' function $f(x)$ is a quadratic function such that from $x = 0$ onwards, the value of $f(x)$ keeps on increasing (in other words, the graph of $f(x)$ goes up from $x = 0$ onwards). Yvonne says that you can never find a positive integer N that makes the quadratic function $g(x) = x^2 - (N + 1) + N$ become "Tullio". Do you think what Yvonne says is true? Please state why."

According to the definition in the task, the quadratic curve must be increasing from the right side of the vertical axis onward in order to be a "Tullio" function. Her instance clearly did not qualified as "Tullio" because a part of the graph (between x = 0 and x = 1.5) decreased and did not concur with the definition. However, she considered it to be a "Tullio" function.

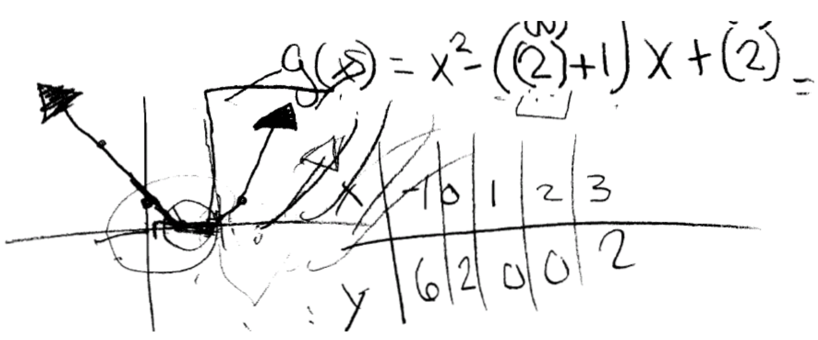

Figure 2: The graphical drawing of instance N = 2 which was classified as "Tullio"

Excerpt 5: Lisa – Classifying a graph by "true means mostly true" conception

Lisa: I thought what it meant was that, from here on [refers to the rising curve on her rough sketch], in a quadratic, that it would go up, which, in my example, when I put 2 for *N*, it didn't. See that's another example, always *true* always *false*, coz right here [circles the turning part of the graph] the part of the graph from it's *not increasing*, so that would make it *false*. But like, *mostly* [traces the rising graph after the turning point] over here, it's *increasing* (Interviewer: Ah, I see). So that's why I said, *no*. That might be where [pause] *all true*.

She noticed that the curve did not match the description of "Tullio" function exactly. However, she made a conscious decision to ignore the part of the graph feature that did not fit. This indicated that her conception in this case was different from the semantics of logical conjunction in the normative conception. Subsequently, she rejected Yvonne's general assertion due to this "counterexample" she found. For a general assertion to be mathematically true, all of instances must be conjunctively satisfied or the assertion is mathematically false. Hence the validation of a general assertion is logically equivalent to a conjunctive statement. Evidently, Lisa was applying the normative conception in rejecting the assertion.

Compared against the normative conception, several deviations could be observed here. Normative conceptions recognized the following notions: the law of middle exclusion (i.e., not true means false and vice versa); the assignment of "true" values to conjunctive statements required all conditions to be "true"; the assignment was monotone and unalterable, and consistent across all mathematical and reasoning contexts. Scrutinized under such a scheme, her conception did not adhere to the law of the middle exclusion whenever she regarded "mostly true" as "true"; the assignment of "mostly true" as "true" defied the logic of conjunctive reasoning; the assignments of "true" varied due to her co-existing and yet logically conflicting conceptions, and were also inconsistent across numerical and graphical contexts. The substantial qualitative deviations noted thus placed her conception furthest away from the normative end in the "distance" model.

Cognitive roles of students' conception of mathematical truth

Students' conceptions of mathematical truth exerted certain influence over their cognitive process at the stage of validating a mathematical assertion. For students like Winnie, Pam, Darren and Sharon, who were highly accustomed to a consistent conception, such a process might be fluent and automatic and hence went unnoticed. However, Polly's and Lisa's cases showed that their conceptions influenced their validation of mathematical statements. In Polly's case, her reasoning of "true" mathematical statements was never conclusive in definitive tone and was easily subject to revisions due to the sense of uncertainty in her conception of mathematical truth. In Lisa's case, her reasoning of "true" mathematical statements was influenced by inconsistent conceptions of mathematical truth, in contrary to the expectation of mathematicians and mathematics educators.

Discussion

Our results suggested that student's conception of mathematical truth set the task parameters and their interpretation of the tasks from which they reasoned towards a conclusion. Their conceptions had different degrees of deviations from the normative conception. The students' conceptions also played a role in their mathematical reasoning processes. In this study, we encountered two students whose conception were much deviant from the normative conception and consequently who struggled to evaluate mathematical statements with their conceptions. Personal conversations with other researchers also revealed anecdotal accounts of struggles faced by the math major students. This led to a question worth pondering: what aspects of their personal learning history contributed to their development of the conceptions?

The influence of High School Mathematics classroom and curriculum

Most of the students in our study (five out of six) were relatively new to the college math curriculum and instructions. Moreover, the topics they were doing such as college algebra and

Calculus level I, were more elaborated versions of what they had learnt during their high school years. Hence it is plausible to infer that much of the development of their conceptions took place during their high school years or even earlier. One might quickly conclude that the high school curricula need to expose students to ideas about mathematical truth as a quick-fix. However, from the lessons we learnt about high school students’ understanding of geometric proofs, we might have to consider carefully how we want to structure the curriculum and instructions (Hanna, 1991). This will require similar research to be conducted at the high school level using instruments that monitor students’ conception of mathematical truth longitudinally.

Students’ Conceptions of Mathematical Truth and Mathematical Proofs

Another consideration that borne out of our study is this: what is the relation between students’ conception of mathematical truth and their conception of mathematical proofs? Both are definitely related to each other in students’ mathematical reasoning processes. The validity of a proof as a written argument rests upon the normative conception since each mathematical statement of a proof has a truth value. It can also be argued the other way round that the conception of mathematical truth is rooted in the production of a valid proof because majority of all mathematically true statements are established epistemologically through producing valid proofs, though the validity of a proof might depend on some implicit social, rather than logical criteria (Hoyles & Küchemann, 2002). Students also believed that proofs play a crucial role in ascertaining mathematically true statements though they might not be competent in producing them (Durand-Guerrier, 2003; Hoyles & Küchemann, 2002).

There are other issues regarding students’ understanding of proofs which might be independent of their conceptions of truth. For example, Duval (1991; 1995) discussed about how the dissociation of students’ cognitive processes from the logical processes in a written geometric proof posed obstacles for students to appreciate the purpose of a geometric proof. In this context, their conceptions of truth seem to matter little in relation to proofs because the truth value of a geometrical assertion is immediately actualized in the diagrams they constructed according to the conditions.

Conclusion

Math education researchers had found that students’ proof schemes and interpretations of logical implication statements varied from the mathematical norms. Our study adds onto these previous findings by illuminating the variations in students’ conception of mathematical truth. The degree of these deviations may be wide as portrayed in the distance model and can thus exert substantial influence in students’ mathematical reasoning.

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